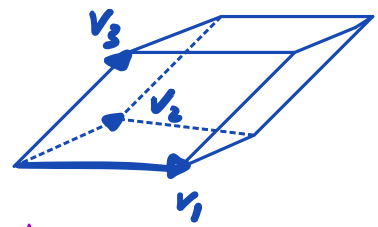
MATH 2028 Differential Forms

Goal: Define differential k-forms in $\mathbb{R}^n$, the exterior derivative and their basic properties

Recall: Given linearly independent $\{v_1, v_2, \dots, v_n\} \subseteq \mathbb{R}^n$

$$\det \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{\text{n-vectors in } \mathbb{R}^n} = \text{signed Volume} \left(\quad \right)$$



n-dimensional parallelopipe

Q: How to find the "signed" volume of a k-dim' parallelopipe in $\mathbb{R}^n$?

(Multi)-linear Algebra

$\mathbb{R}^n$: Standard basis $\{e_1, \dots, e_n\}$ where $e_i = (0, \dots, 0, \overset{i^{th} \text{ coordinate}}{1}, 0, \dots 0)$

$\updownarrow$ dual $\qquad\qquad \updownarrow$ dual

$(\mathbb{R}^n)^*$: dual basis $\{dx_1, \dots, dx_n\}$ s.t. $\boxed{dx_i(e_j) = \underbrace{\delta_{ij}}_{\text{"Kronecker delta"}}}$

Any linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ has the form

$$\phi = a_1 \, dx_1 + a_2 \, dx_2 + \cdots + a_n \, dx_n$$

We generalize this to multi-linear functions.

Given $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we define

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} : \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{k \text{ copies}} \longrightarrow \mathbb{R}$$

as $(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k})(v_1, v_2, \ldots, v_k)$

$$= \det \begin{pmatrix} dx_{i_1}(v_1) & \cdots & dx_{i_1}(v_k) \\ \vdots & & \vdots \\ dx_{i_k}(v_1) & \cdots & dx_{i_k}(v_k) \end{pmatrix} \quad k \times k \text{ matrix}$$

which is a k-linear alternating map on $\mathbb{R}^n$.

FACT: $\{ dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \}_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}$

forms a basis of the vector space of k-linear alternating maps on $\mathbb{R}^n$, denoted by $\Lambda^k(\mathbb{R}^n)^*$

Hence,

$$\dim \Lambda^k(\mathbb{R}^n)^* = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

E.g.) $n=3$, $k=2$ A basis for $\Lambda^2(\mathbb{R}^3)^*$ is given by

$$\{dx_1 \wedge dx_2,\ dx_1 \wedge dx_3,\ dx_2 \wedge dx_3\}$$

Wedge Product

The following notion of "wedge product" $\wedge$ generalizes the cross product $\times$ of vectors in $\mathbb{R}^3$.

$$(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell})$$

$$:= dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}$$

Remember that: (alternating)
$$\begin{cases} dx_i \wedge dx_i = 0 \\ dx_i \wedge dx_j = -dx_j \wedge dx_i \end{cases}$$

Extending linearly, we get a bilinear map

$$\begin{array}{ccccc} \wedge : & \Lambda^k(\mathbb{R}^n)^* \times \Lambda^\ell(\mathbb{R}^n)^* & \longrightarrow & \Lambda^{k+\ell}(\mathbb{R}^n)^* \\ & (\omega, \eta) & \longmapsto & \omega \wedge \eta \end{array}$$

which is skew-commutative:

$$\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega$$

and associative: $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$

E.g.) $\left.\begin{aligned} \omega &= a_1\, dx_1 + a_2\, dx_2 \\ \eta &= b_1\, dx_1 + b_2\, dx_2 \end{aligned}\right\}$ in $\Lambda^1(\mathbb{R}^2)^*$

$$\begin{aligned}
\omega \wedge \eta &= (a_1\, dx_1 + a_2\, dx_2) \wedge (b_1\, dx_1 + b_2\, dx_2) \\
&= a_1 b_1\, \overset{=0}{dx_1 \wedge dx_1} + a_1 b_2\, dx_1 \wedge dx_2 \\
&\qquad + a_2 b_1\, dx_2 \wedge dx_1 + a_2 b_2\, \overset{=0}{dx_2 \wedge dx_2} \\
&= \underbrace{(a_1 b_2 - a_2 b_1)}_{\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}\, dx_1 \wedge dx_2
\end{aligned}$$

same up to a sign

Differential Forms on $\mathbb{R}^n$

Notation: $I = (i_1, i_2, \ldots, i_k)$ increasing k-tuple.

A **differential k-form** on $\mathbb{R}^n$ is an expression

$$\omega = \sum_{I = (i_1, \ldots, i_k)} f_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where f_I are smooth functions on (subset of) $\mathbb{R}^n$.

E.g.) 0-forms are just functions

1-forms: $\omega = f_1\, dx_1 \wedge \cdots \wedge f_n\, dx_n$

n-forms: $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$

Remark: We can take wedge product of differential forms pointwise as before. More importantly, we have a way to "differentiate" differential forms

Notation: $\mathcal{A}^k(\mathcal{U}) = \left\{ \begin{array}{c} \text{differential } k\text{-forms} \\ \text{on } \mathcal{U} \subseteq \mathbb{R}^n \end{array} \right\}$

Def^n: There exists an exterior derivative

$$d : \mathcal{A}^k(\mathcal{U}) \longrightarrow \mathcal{A}^{k+1}(\mathcal{U})$$

s.t. (1) d is linear

(2) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

where $\omega \in \mathcal{A}^k(\mathcal{U})$, $\eta \in \mathcal{A}^\ell(\mathcal{U})$

(3) $d^2 = d \circ d = 0$

(4) $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \quad \forall$ function f

Examples: (1) $df = f'(x)\, dx$, $\forall f : \mathbb{R} \to \mathbb{R}$

(2) $\omega = y\,dx + x\,dy \in \mathcal{A}^1(\mathbb{R}^2)$

$d\omega = d(y\,dx) + d(x\,dy) = dy \wedge dx + dx \wedge dy = 0$

(3) $\omega = -y\,dx + x\,dy \in A^1(\mathbb{R}^2)$

$$d\omega = -dy \wedge dx + dx \wedge dy = 2\,dx \wedge dy$$

(4) $\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \in A^1(\mathbb{R}^2 \setminus \{0\})$

$$d\omega = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] dx \wedge dy$$

$$= \left[\frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\right] dx \wedge dy = 0$$

FACT: d generalize the notion of grad, curl and div.

Given a function f on $U \subseteq \mathbb{R}^n$.

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n$$

Hence, writing the R.H.S. in terms of the basis $\{dx_1, dx_2, \ldots, dx_n\}$ of $A^1(U)$, we have

$$df = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) = \nabla f$$

Therefore, $d : A^0(U) \to A^1(U)$ is the "gradient" differential operator on functions.

Given a vector field $F: \mathcal{U} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, s.t. $F = (F_1, F_2, F_3)$ in components, if we identify it with the 2-form:

$$\omega = F_1\, dy \wedge dz - F_2\, dx \wedge dz + F_3\, dx \wedge dy$$

Then,

$$\begin{aligned} d\omega &= dF_1 \wedge dy \wedge dz - dF_2 \wedge dx \wedge dz \\ &\quad + dF_3 \wedge dx \wedge dy \\ &= \frac{\partial F_1}{\partial x}\, dx \wedge dy \wedge dz - \frac{\partial F_2}{\partial y}\, dy \wedge dx \wedge dz \\ &\quad + \frac{\partial F_3}{\partial z}\, dz \wedge dx \wedge dy \\ &= \underbrace{\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right)}_{= \operatorname{div} F} dx \wedge dy \wedge dz \end{aligned}$$

Similar calculation also works for vector fields in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Finally, we can also recover the curl operator using the exterior derivative d.

Let $\omega = P\,dx + Q\,dy \in \mathcal{A}^1(\mathbb{R}^2)$. Then

$$
\begin{aligned}
d\omega &= dP \wedge dx + dQ \wedge dy \\
&= \frac{\partial P}{\partial y}\, dy \wedge dx + \frac{\partial Q}{\partial x}\, dx \wedge dy \\
&= \Bigg(\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\substack{\text{2 dim'l curl} \\ \text{of } F = (P,Q)}} \Bigg)\, dx \wedge dy
\end{aligned}
$$

Let $\omega = F_1\,dx + F_2\,dy + F_3\,dz \in \mathcal{A}^1(\mathbb{R}^3)$. Then

$$
\begin{aligned}
d\omega &= dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \\
&= \frac{\partial F_1}{\partial y}\, dy \wedge dx + \frac{\partial F_1}{\partial z}\, dz \wedge dx \\
&\quad + \frac{\partial F_2}{\partial x}\, dx \wedge dy + \frac{\partial F_2}{\partial z}\, dz \wedge dy \\
&\quad + \frac{\partial F_3}{\partial x}\, dx \wedge dz + \frac{\partial F_3}{\partial y}\, dy \wedge dz \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) dz \wedge dx \\
&\qquad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \wedge dy
\end{aligned}
$$

whose components are equal to curl (F_1, F_2, F_3).

Pullback of differential forms

Given a C^∞ map $g: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$, we can use it to "pullback" differential forms:

$$g^*: A^k(\mathbb{R}^n) \longrightarrow A^k(U)$$

<u>0-forms:</u> $g^* f := f \circ g \quad \forall f \in A^0(\mathbb{R}^n)$

<u>1-forms:</u> Write in components $g = (g_1, \dots, g_n)$, define

$$g^*(f_1 dx_1 + \cdots + f_n dx_n)$$

$$= (f_1 \circ g) dg_1 + \cdots + (f_n \circ g) dg_n$$

<u>k-forms:</u> $g^*\left(\sum_I f_I dx_I\right) = \sum_I (f_I \circ g) dg_I$

Let us illustrate by some examples.

<u>Examples:</u>

(1) $g: \mathbb{R} \to \mathbb{R}$, $g^*(f(x) dx) = f(g(u)) g'(u) du$

(2) $g: \mathbb{R} \to \mathbb{R}^2$. $g(t) = (\cos t, \sin t)$

$$g^*(-y dx + x dy) = -\sin t \, d(\cos t) + \cos t \, d(\sin t)$$

$$= (\sin^2 t + \cos^2 t) dt = dt$$

Thm: $g^*(d\omega) = d(g^*\omega)$. $\forall\, \omega \in A^k(\mathbb{R}^n)$

Proof: $\underline{k=0}$: let $f \in A^0(\mathbb{R}^n)$.

$$d(g^*f) = d(f\circ g)$$

$$= \sum_{j=1}^{m} \frac{\partial}{\partial u_j}(f\circ g)\, du_j$$

(Chain Rule) $$= \sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\circ g\right)\frac{\partial g_i}{\partial u_j}\right] du_j$$

$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\circ g\right)\left[\sum_{j=1}^{m}\frac{\partial g_i}{\partial u_j}\, du_j\right]$$

$$= \sum_{i=1}^{n} g^*\frac{\partial f}{\partial x_i}\cdot g^* dx_i$$

$$= g^*\left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\, dx_i\right) = g^*(df)$$

$\underline{k>0}$: By linearity, it suffices to check

$$g^*(d(f\, dx_I)) = g^*(df \wedge dx_I) = g^*(df)\wedge g^*(dx_I)$$

$$= d(g^*f)\wedge dg_I = d(g^*f\, dg_I)$$

$$= d(g^*(f\, dx_I))$$

$\square$